Traversable Wormholes

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Abstract

The *wormhole* is a hypothetical feature of spacetime that connects distant regions of the universe (or even distinct universes) together. While the concept has been around since the early years of general relativity theory, a thorough analysis of their feasibility for human traversal was not attempted until the mid 1980s. It turns out that traversable wormholes are possible under the laws of GR if one allows so-called *exotic matter* with negative energy, which is usually considered unrealistic but not necessarily impossible. One common objection is that wormholes can link regions that are distant in time as well as space, which has the potential to introduce paradoxes. This essay gives a mathematical description of some a hypothetical wormhole and reviews the literature regarding the aforementioned issues.

Natural units with $c, G = 1$ are used throughout.

1 Introduction

The idea of a wormhole was first considered by Ludwig Flamm [1] less than a year after Einstein published his theory of General Relativity (GR). Einstein's field equation

$$R_{ij} - \frac{1}{2} R^k{}_k g_{ij} = 8\pi T_{ij} \tag{1}$$

places no restriction on the topology of spacetime, it simply relates the geometry with energy and momentum. Flamm realised that it therefore seems just as valid to base spacetimes on arbitrary differentiable 4-manifolds as it is to use the usual $\mathbb{R}^4$, which allows for the insertion of topological "handle" joining two distant parts of the universe. By adjusting the metric on the handle, one can then make the distances measured via the two routes different, giving us what has come to be known as a wormhole.

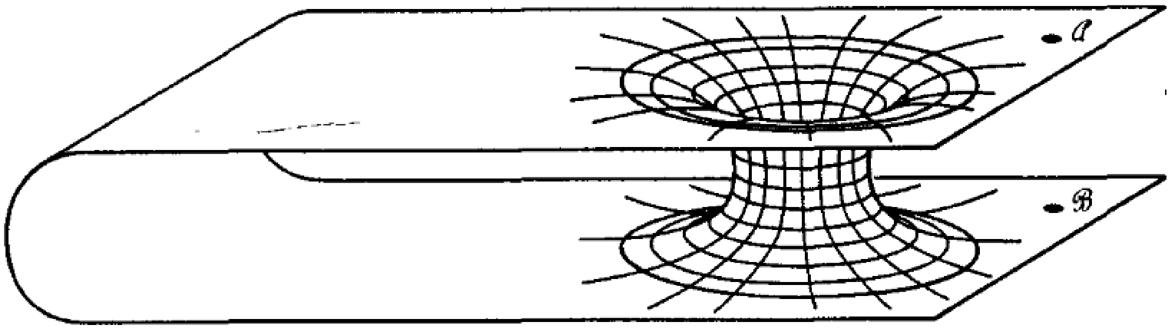

Figure 1: A wormhole between two distant locales of the same universe. In the vicinity of the wormhole, the fact that there is a "long way around" is irrelevant; the geometry is the same as in a wormhole between two different universes. Note that the folding of the surface is an extrinsic property of this particular embedding that allows us to depict the situation conveniently; the actual geometry is flat in that region. *Image from [4].*

2 The Einstein-Rosen Bridge

An explicit example of a wormhole metric was investigated by Einstein and Rosen in their 1935 paper [2] and came to be known as the Einstein-Rosen bridge. The Einstein-Rosen bridge connects two distinct universes; but for most purposes, we can easily adapt an inter-universe wormhole metric by "joining up" the two universes and smoothing out the metric in between (see Figure 1).

Interestingly, the metric used was exactly that of the Schwartzchild solution, which describes the spacetime outside a spherically symmetric mass m. In the usual spherical coordinates, the Schwartzchild metric

$$g = -\left(1 - 2m/r\right) dt^2 + \frac{1}{1 - 2m/r} dr^2 + r^2 d\Omega^2 \tag{2}$$

is singular at $r = 2m$, and is often only considered on $(t, r, \theta, \varphi) \mid r > 2m \simeq \mathbb{R}^2 \times S^2$. (In the case where the radius of the generating mass is smaller than $r = 2m$, we have an event horizon at $r = 2m$ and $r < 2m$ represents the interior of a black hole.) By changing the radial coordinate to $u = \sqrt{r - 2m}$ and reflecting the metric to negative u, one finds

$$g = -\frac{u^2}{u^2 + 2m} dt^2 + 4\left(u^2 + 2m\right) du^2 + \left(u^2 + 2m\right)^2 d\Omega^2. \tag{3}$$

This is now well-defined and smooth for all values of (t, u, θ, φ), so the singularity at $u = 0$ in the old coordinates is simply a property of those coordinates. (There is a genuine singularity at $r = 0$, but our new coordinates do not cover this region.) We can now interpret the metric as a wormhole (see Figure 2 for a visualisation). Noting that $dr = 2u du$, we find that in the limit $u \to \pm\infty$ where we take $u = \pm\sqrt{r - 2m}$ the metric tends towards the flat metric

$$\eta = -dt^2 + dr^2 + r^2 d\Omega^2; \tag{4}$$

i.e. we have two universes that asymptotically look like Minkowski space, joined together at the *throat* $r = 2m$. Because we actually have two separate coordinate charts for positive and negative u, we cannot use these coordinates to draw conclusions about geodesics crossing $u = 0$. (The fact that $\det g = 0$ at $u = 0$ is also a hint that things are not as regular there as they seem.) A complete analysis [4, Ch 31] using the Kruskal-Szekeres coordinates (which cover the entire extended geometry, including both universes and both interior regions) shows that there are in fact no geodesics that pass from one universe to the other. Any geodesic that crosses the horizon ends up in the $r < 2m$ portion of the geometry and ultimately terminates at the singularity $r = 0$; so we say that this wormhole is

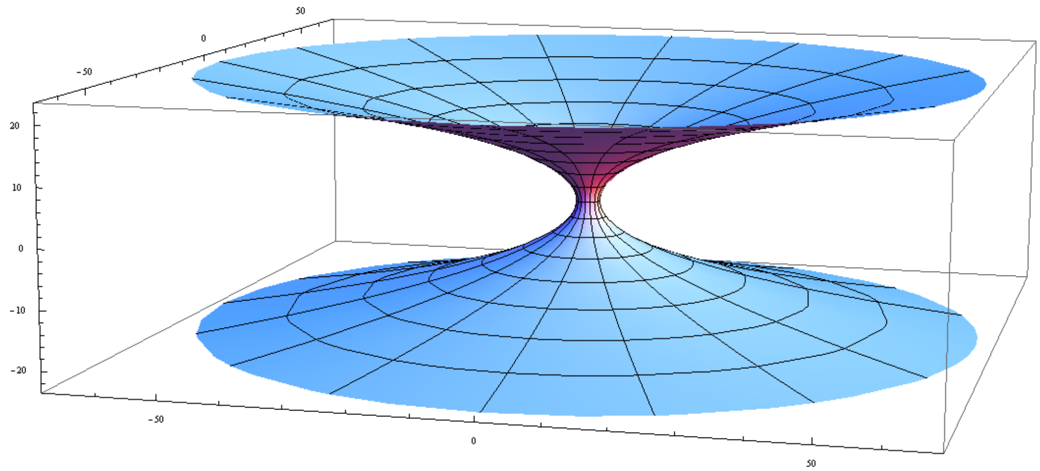

Figure 2: Embedding diagram of Einstein-Rosen bridge. We take a slice of the spacetime $t = 0$, $\theta = \pi/2$ which gives a 2-manifold with metric $4\left(u^2 + 2m\right) du^2 + \left(u^2 + 2m\right)^2 d\varphi^2$, which is then isometrically embedded into $\mathbb{R}^3$ with cylindrical coordinates via the map $(u, \varphi) \mapsto \left(u^2 + 2m, \varphi, \sqrt{8mu}\right)$. This figure uses $m = 1$, $u \in (-8, 8)$, $\varphi \in [0, 2\pi)$. Angles and lengths measured on the surface are the same as those in the full spacetime.

not traversable. Two travelers from different universes could briefly meet inside the event horizon, but they could never send a message back and would be promptly crushed.

3 The Morris-Thorne Wormhole

The term *wormhole* was coined by Misner and Wheeler in 1957 [5], and the concept quickly became a popular method of travel in science fiction - they are a convenient device to allow rapid interstellar travel without neglecting the universal speed limit c. Despite this, it was not until 1985 that real scientific interest arose in the possibility of traversable wormholes. While writing his novel *Contact*, Carl Sagan consulted physicists Kip Thorne and Michael Morris for help depicting a believable wormhole, and they found a remarkably simple example of a traversable wormhole consistent with almost all of GR [3]. They started with the assumption of spherical symmetry and two asymptotic flat regions (both possessed by the Einstein-Rosen bridge discussed above), and constructed a metric with geodesics passing from universe to the other in reasonable timescales.

Similarly to the Einstein-Rosen bridge, the Morris-Thorne metric uses coordinates t, l, φ, θ on $\mathbb{R}^2 \times S^2$. The coordinate l will measure proper radial distance from the throat (with $l < 0$ in one universe), but it is more convenient to construct the metric using the coordinate r where $2\pi r$ is the circumference of a circle $\varphi \mapsto (t_0, r, \varphi, \pi/2)$. Assuming spherical and time-translation symmetry, we write down the general Morris-Thorne metric

$$g = -e^{2\Phi} dt^2 + \frac{1}{1 - b/r} dr^2 + r^2 d\Omega^2 \tag{5}$$

where Φ, b are adjustable functions of r only. To achieve asymptotic flatness, we need this to converge to η as $r \to \infty$; so we have constraints

$$\lim_{r\to\infty} \Phi(r) = 0 \qquad \lim_{r\to\infty} \frac{b(r)}{r} = 0. \tag{6}$$

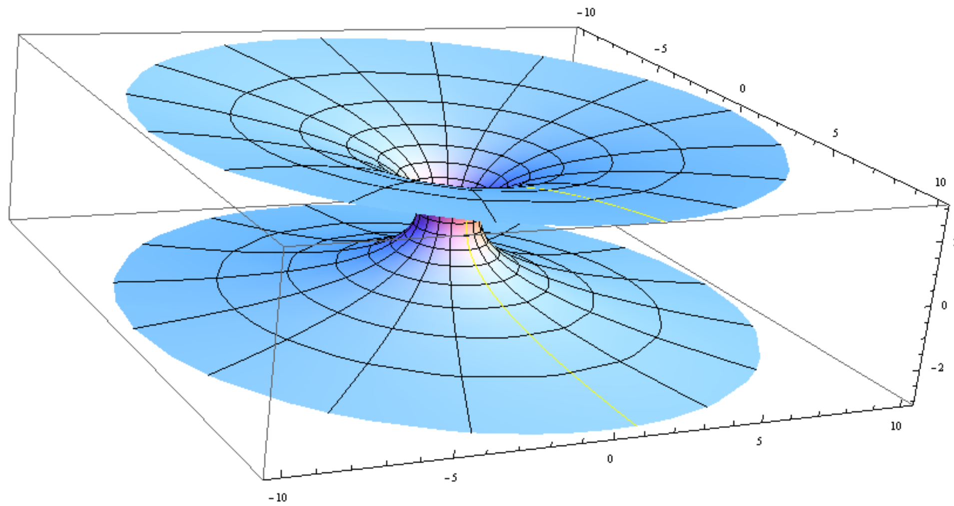

Figure 3: Embedding diagram for Morris-Thorne wormhole with $\Phi(r) = 0,\ b(r) = 1/r$. Slice taken is $t = 0, \theta = \pi/2$. Cylindrical embedding map is $(r,\varphi) \mapsto (r = \cosh z, \varphi, z)$ where $z(r) = \ln\left(r + \sqrt{r^2-1}\right)$. Yellow line shows a geodesic passing from one universe to another.

We can recover the Schwartzchild metric by setting $b(r) = 2m, \Phi(r) = \frac{1}{2}\ln(1-2m/r)$. We will investigate the simple but fruitful example $\Phi(r) = 0,\ b(r) = b_0^2/r$, which yields $l^2 = r^2 - b_0^2$ and therefore in coordinates (t,l,θ,φ) the metric can be written

$$g = -dt^2 + dl^2 + \left(l^2 + b_0^2\right) d\Omega^2. \tag{7}$$

Figure 3 shows that the geometry appears qualitatively similar to that of the Einstein-Rosen bridge. In this case, however, we have a valid global coordinate chart with no degeneracies, so what we see is the genuine spacetime. This means that the trajectories $\tau \mapsto x(\tau) = (\gamma\tau, l_0 - \gamma v\tau, 0, 0),\ \gamma = \left(1-v^2\right)^{-1/2}$ are geodesics that start in one universe at radial position l_0 and travel through the wormhole to the other universe. To see this, note that g_{tt}, g_{ll} have no dependence on t, l, so all the relevant Christoffel symbols vanish and therefore the acceleration is simply $-\gamma(\partial_t v)\partial_l = 0$. Assuming no forces will act on a particle following this trajectory, we can therefore send matter through the wormhole. For it to truly be deserving of the name *traversable* we would also want a human to be able to comfortably travel through it in a reasonable amount of time; so we have a few more requirements. Firstly, the gravitational and tidal accelerations felt by the traveler should be small so that the traveler is not crushed or spaghettified. For a traveler freefalling on a geodesic, the gravitational acceleration is zero, so there is no issue here. The relative acceleration between x and a geodesic separated from it by a small vector δx is approximately [4, Ch 11]

$$R_{\dot{x}}(\delta x) = R(\delta x, \dot{x})\,\dot{x} = R^i{}_{jkl}\dot{x}^j \delta x^k \dot{x}^l. \tag{8}$$

In this case, there are only a few non-zero curvature components. Switching to the local reference frame of a stationary observer (i.e. the orthonormal frame $E_i = |g_{ii}|^{-1/2}\partial_i$), they are [3]

$$R_{\theta\varphi\theta\varphi} = -R_{l\theta l\theta} = -R_{l\varphi l\varphi} = \frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \tag{9}$$

and the other components that can be derived from these via the Riemann tensor's symmetries. The velocity of the traveler in this frame is

$$\dot{x} = \gamma(\partial_t - v\partial_l) = \gamma(E_t - vE_l) \tag{10}$$

and therefore the tidal acceleration is

$$R^{\theta}{}_{l\theta l}\left(\dot{x}^{l}\right)^{2}\delta x^{\theta}E_{\theta}+R^{\varphi}{}_{l\varphi l}\left(\dot{x}^{l}\right)^{2}\delta x^{\varphi}E_{\varphi}=-\frac{b_0^2}{\left(b_0^2+l^2\right)^2}\gamma^2 v^2\left(\delta x^{\theta}E_{\theta}+\delta x^{\varphi}E_{\varphi}\right). \tag{11}$$

This has a maximum at $l=0$, where we find (for someone 2m tall laying perpendicular to the radial direction, which maximises the tidal force)

$$a_{\text{tide}} = 2\text{m}\,\gamma^2 v^2 \tag{12}$$

and therefore we can decrease the tidal acceleration below any given threshold by decreasing the velocity.

We now have a satisfactory geometry for a wormhole, so the outstanding question is to ask what matter distribution generates this geometry. Working in the static orthonormal frame as before and starting from Equation 9, we find the non-zero components of the Ricci tensor are

$$R_{\theta\theta} = R_{\theta\varphi\theta\varphi}+R_{\theta l\theta l}=0 \tag{13}$$

$$R_{ll} = R_{l\theta l\theta}+R_{l\varphi l\varphi}=-2\frac{b_0^2}{\left(b_0^2+l^2\right)^2} \tag{14}$$

$$R_{\varphi\varphi} = R_{\varphi\theta\varphi\theta}+R_{\varphi l\varphi l}=0. \tag{15}$$

The scalar curvature is therefore $R=R^k_k=-2b_0^2/\left(b_0^2+l^2\right)^2$ and so using Equation 1 we find we have a diagonal stress-energy tensor:

$$-T_{tt}=-T_{ll}=T_{\varphi\varphi}=T_{\theta\theta}=\frac{1}{8\pi}\frac{b_0^2}{\left(b_0^2+l^2\right)^2}. \tag{16}$$

4 Exotic Matter

We've now struck our first real problem with the Morris-Thorne wormhole: the energy density T_{tt} is negative. While we have only studied a particular example here, the full analysis [3] shows that for any choice of b,Φ satisfying (6), there is some observer who sees $T_{tt}<0$, so this is a generic feature of Morris-Thorne wormholes. Intuitively, the space needs to be negatively curved in order to smoothly transition from the throat to the asymptotically flat universe, so the field equations necessitate *exotic matter* with negative mass to hold it open. While this matter is distributed across all of space (though concentrated near the wormhole) in our simple example, one can find b,Φ that give safely traversable wormholes such that T is zero outside of a thin spherical shell [3].

Because Morris and Thorne assumed spherical symmetry in the geometry, we have spherical symmetry in the distribution of exotic matter; so anyone traversing the wormhole will necessarily pass through it. While the interactions of the exotic matter with other matter is unknown, it is likely that this would be undesirable. By discarding spherical symmetry, Matt Visser showed [6] the existence of a wormhole metric where travelers do not come in contact with the exotic matter, and in addition do not feel any gravitational or tidal forces.

While the field equations and principle of geodesic motion are perfectly consistent in a universe with negative energy, it is conventional to impose some restrictions on the energy in order to get realistic results. Normally one assumes at least the weak energy condition, which states that the energy density is always measured as non-negative by every observer, i.e. $T(V,V)\geq 0$ for every timelike vector field V. This is more important than just getting rid of the troublesome concept of negative mass - much of black hole thermodynamics relies on the weak energy condition [1]; so many would dismiss solutions of (1) that violate it as non-physical.

All is not lost, however - quantum field theory (QFT) offers a tantalising possibility in the form of the Casimir effect. If two conducting planes are placed parallel to each other at a distance L, the boundary conditions suppress electromagnetic field modes with wavelengths $\lambda > 2L$. As the plates are brought closer together, these excluded modes increase in energy, and one finds that the vacuum energy between the plates is lower than that of free space. Because the vacuum energy of free space is conventionally taken as the zero point, this means that there is essentially a negative energy density between the plates. The Casimir effect has been experimentally validated [7], and is therefore very good evidence that the weak energy condition does not hold in general. While the Casimir effect itself is almost certainly not practical for supporting a wormhole, it hints that QFT (or better, a unified theory of quantum gravity) can admit negative energy densities. Whether or not usable exotic matter exists in the universe (or can be manufactured) is another question entirely - there is certainly no empirical evidence for such materials.

5 The Chronology Problem

One common argument against the existence of wormholes is that they introduce time travel paradoxes. One can easily create a single-universe geometry (as in Figure 1) with a wormhole whose two exits are attached at distant points in time. One can find closed timelike curves (CTCs; i.e. periodic particle trajectories) that pass through the wormhole to the past, and then travel back to the future the long way. This introduces the possibility of a given initial condition violating itself when it propagates around the loop, as in the classic example of a man killing his own grandfather. This is not, however, particular to wormholes: there are many known spacetimes with CTCs, often with otherwise reasonable conditions (i.e. no violation of energy conditions) [9]

Because a full theory of quantum gravity has not yet been developed, the chronology problem has a number of resolutions [1], from conservative to radical:

- Conject that our particular universe is globally hyperbolic, implying no CTCs and therefore that Morris-Thorne wormholes do not exist. (It can be shown [10] that in the case of classical GR, *any* Morris-Thorne wormhole can be manipulated into a configuration that contains a CTC.) This is often known as the boring physics conjecture.

- Hawking's chronology protection conjecture: assume that quantum mechanics will (via some yet to be determined mechanism) intervene to forbid the existence of macroscopic CTCs. While some have attempted to prove/disprove this conjecture using the theory of semiclassical quantum gravity, these analyses are possibly outside the regime where such an approximation is valid [11]; so a full theory of quantum gravity is required before this can really be investigated. Depending on the nature of the mechanism, this could possibly preclude the existence of wormholes, but may admit them as long as they do not contain CTCs.

- The Novikov consistency conjecture: accept time travel, and simply require that it be consistent, i.e. conjecture that the universe will only accommodate globally consistent solutions. From one point of view, this does not modify our laws of physics at all: a non-consistent time loop would result in a discontinuity of the metric or some field, meaning it would no longer be a solution to the governing PDEs. However, this may remove the locality of the physical laws. We can take the state (i.e. the spatial topology, the metric and any auxiliary fields) on one local spacelike hypersurface $\{t_0\} \times U$ and use it as initial conditions to find a unique local solution on $[t_0, t_0 + \epsilon) \times U$. The Novikov conjecture implies that in a spacetime where $\{t_0 + \tau\} \times U$ connects back up with $\{t\} \times U$ via geodesics, such a solution may not in fact be correct if it does not extend to a smooth *global* solution.

- The radical rewrite conjecture: discard altogether the requirement of a single consistent timeline and allow multiple branching timelines. This requires a fundamental modification of our theory, even at the macroscopic level. One possible model for this is the extension of general relativity to a certain class of non-Hausdorff manifolds, allowing a "single-sheeted" universe to split into two [1].

The last two conjectures would admit wormholes - under the consistency conjecture they would be exactly as described earlier, but after radical rewrite they would take somewhat of a different form to fit the new model.

6 Conclusions

Traversable wormholes certainly exist as solutions to the classical equations of general relativity *if and only if* we allow the violation of the energy conditions. The exotic matter required to do so is unlike anything yet discovered or manufactured, but is potentially possible due to quantum effects. The relationship between wormholes and time travel depends entirely on which of the four chronology conjectures is true - quantum gravity may intervene to rule them out entirely, allow them while barring time travel or place no restriction upon them at all. Ultimately, the feasibility of traversable wormholes cannot be determined until we either have a validated theory of quantum gravity, or direct empirical evidence.

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