Differential Geometry and PDEs

Anthony Carapetis
Supervisor: Gilbert Weinstein

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Abstract

(Semi-)Riemannian geometry and partial differential equations (PDEs) are the mathematical theory underlying most of modern physics, and in particular the theory of General relativity. This paper is a brief introduction to the concepts of differential geometry and elliptic partial differential equations, with an example application to Plateau's problem in minimal surfaces; though many of the same techniques have more recently been applied to problems related to general relativity. The first section is a primer for differential geometry, the theory of general spaces where the concept of differentiation can be defined. Next we discuss the theory of PDEs, which arise frequently in differential geometry and applications. We approach PDEs via the calculus of variations, including a discussion of the Euler-Lagrange equations for an action and the direct method for proof of the existence of a weak solution. While our exposition only covers Laplace's equation, the same methods extend to a wide class of PDEs. We briefly touch on the theory of regularity for elliptic PDEs. Finally, all this machinery is applied to the solution of Plateau's problem.

Contents

1 Smooth Manifolds

Smooth manifolds are a generalisation of Euclidean space that retain the concepts of differentiability and vector fields. We start with a topological space, and then make local identifications with $\mathbb{R}^n$ that we use to define differentiation. These local identifications are just coordinate systems, known as charts.

1.1 The basics

Definition 1. An m-dimensional CHART on a topological space M is a homeomorphism ξ from an open subset of M to an open subset of $\mathbb{R}^m$. The term "chart" will sometimes be used to refer to the image in $\mathbb{R}^m$ when this is not ambiguous.

A smooth atlas $\mathscr{A}$ on M is a collection of charts on M such that every point in M is in the domain of at least one chart in $\mathscr{A}$, and for every pair ξ, η of charts in $\mathscr{A}$ with overlapping domains, the map $\xi \circ \eta^{-1}$ is smooth (as a function between subsets of $\mathbb{R}^m$).

A smooth atlas $\mathscr{A}$ on M is COMPLETE or MAXIMAL if there is no larger atlas $\mathscr{B} \supsetneq \mathscr{A}$.

An m-dimensional MANIFOLD (or m-MANIFOLD) $(M, \mathscr{A})$ is a topological space M equipped with a complete atlas $\mathscr{A}$ of m-dimensional charts.

Any atlas has a unique extension to a complete atlas (order the set of all atlases on M by inclusion and apply Zorn's lemma).

Definition 2. A function $f : M \to \mathbb{R}$ is SMOOTH if for every point $p \in M$, there is a chart ξ with domain $U \ni p$ such that $f \circ \xi^{-1}$ is smooth.

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle \xi}\downarrow & \nearrow_{f\circ\xi^{-1}} & \\ \mathbb{R}^m & & \end{array}$$

The set of all smooth functions on M is denoted by $\mathscr{F}(M)$ or $C^\infty(M)$, and is an associative $\mathbb{R}$-algebra under the obvious operations of pointwise addition, scalar multiplication and pointwise multiplication.

A function $f : M \to N$ is SMOOTH if for every point $p \in M$ there is a chart ξ on $U \ni p$ and a chart η on $V \ni f(p)$ such that $\eta \circ f \circ \xi^{-1}$ is smooth.

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathbb{R}^m & \xrightarrow[\eta\circ f\circ\xi^{-1}]{} & \mathbb{R}^n \end{array}$$

A bijection f between manifolds is a DIFFEOMORPHISM if both f and f^{-1} are smooth. If $\mathbb{R}$ is considered as a manifold (with the identity map as a global chart) then these two notions of smoothness agree.

1.2 Vectors and Vector Fields

Definition 3. A DERIVATION on an $\mathbb{R}$-algebra A is an $\mathbb{R}$-linear map $D : A \to A$ such that $D(ab) = aDb + bDa$.

The requirement of the Leibniz product rule means derivations are like differential operators; the one-dimensional derivative is a derivation on the algebra of smooth functions $\mathbb{R} \to \mathbb{R}$.

Definition 4. A TANGENT VECTOR at $p \in M$ is an $\mathbb{R}$-linear function $v : \mathscr{F}(M) \to \mathbb{R}$ such that $v(fg) = f(p)\,vg + g(p)\,vf$. The space of all tangent vectors at p is denoted by T_pM and is called the TANGENT SPACE. The tangent space is an m-dimensional vector space, with basis

$$\left\{ \partial_i|_p \equiv \frac{\partial}{\partial x^i}|_p : i \in \{1 \dots n\} \right\}$$

where the basis vectors are defined by

$$\frac{\partial}{\partial x^i}|_p \phi = \partial_i \left(\phi \circ \xi^{-1} \right) (\xi(p))$$

where the ∂_i is the usual partial derviative of functions on $\mathbb{R}^n$. The dual basis to $\{\partial_i\}$ (which spans the COTANGENT SPACE $T_p^*M = (T_pM)^*$) is denoted by $\{dx^i\}$, and the components of $v \in T_pM$ in the co-ordinate system ξ are $v^i = dx^i(v)$.

Definition 5. The TANGENT BUNDLE is the fibre bundle formed by all the tangent spaces $TM = \bigsqcup_{p \in M} T_pM$, and is a $2m$-dimensional manifold when equipped with charts

$$\xi^*(p, v) = \left(x^1(p) \dots x^m(p), v^1 \dots v^m \right)$$

for charts $\xi = \left(x^1 \dots x^m \right)$ on M. The projection from a vector to its base point will be denoted by $\pi : TM \to M$.

A VECTOR FIELD is a smooth section of the tangent bundle; i.e. a smooth function $X : M \to TM$ such that $\pi \circ X = \mathrm{id}$. A vector field can also be thought of as an operator on $\mathscr{F}(M)$, by applying pointwise: $(Xf)(p) = X_p f(p)$. Vector fields are derivations on $\mathscr{F}(M)$, and every derivation on $\mathscr{F}(M)$ is a vector field. The space of vector fields on M is denoted by $\mathscr{X}(M)$ or $\Gamma(TM)$, and is an $\mathscr{F}(M)$-module (where $(fX)(p) = f(p)X_p$).

Similarly, a ONE-FORM is a smooth section of the COTANGENT BUNDLE $T^*M = \bigsqcup_{p \in M} T_p^*M$; the space of one-forms on M is the $\mathscr{F}(M)$-module $\mathscr{X}^*(M)$.

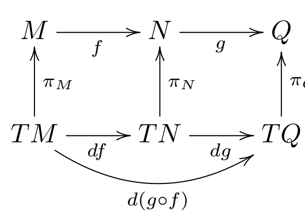
The co-ordinate vector fields $\{\partial_i\}$ are a basis for $\mathscr{X}(M)$ over $\mathscr{F}(M)$. Likewise, the one-forms $\{dx_i\}$ are a basis for $\mathscr{X}^*(M)$ over $\mathscr{F}(M)$.

The LIE BRACKET of two vector fields X and Y is $[X, Y] = XY - YX$, where the juxtaposition of vector fields denotes composition of the corresponding derivations.

The DIFFERENTIAL at $p \in M$ of a smooth map $f \in \mathscr{F}(M)$ is the cotangent vector (i.e. a linear map $df_p : T_pM \to \mathbb{R}$) given by $df_p(v) = v(f)$. Likewise, the differential (or PUSHFORWARD) at $p \in M$ of a smooth map $f : M \to N$ is a linear map between tangent spaces $df_p : T_pM \to T_{f(p)}N$ given by

$$\forall \phi \in \mathscr{F}(N) \quad df_p(v)(\phi) = v(\phi \circ f).$$

Bundling the differentials from all points in the manifold produces an $\mathbb{R}$-linear map df between tangent bundles; which in the case of $N = \mathbb{R}$ is a one-form. The differential distributes across function composition; i.e. $d(g \circ f) = dg \circ df$.



In terms of co-ordinates $\xi = \left(x^1 \dots x^m \right)$ on M and $\eta = \left(y^1 \dots y^n \right)$ on N, the pushforward of a vector tangent to M is

$$df\left(v^i \frac{\partial}{\partial x^i} \right) = \left(\frac{\partial \varphi^j}{\partial x^i} v^i \right) \frac{\partial}{\partial y^i}$$

where φ is the map between charts corresponding to f.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ \mathbb{R}^m & \xrightarrow[\varphi]{} & \mathbb{R}^n \end{array}$$

We therefore see that the matrix representation of the differential is the familiar Jacobian matrix from multivariable calculus.

Definition 6. A CURVE in M is a smooth map $\gamma : I \to M$, where I is an interval in the real line. The VELOCITY of γ at t (or at $p = \gamma(t)$) is the tangent vector $\dot{\gamma}(t) \in T_p M$ given by

$$\dot{\gamma}(t) = d\gamma_t \left(\frac{d}{dt} \right)$$

where d/dt is the ordinary derivative operator on $\mathscr{F}(I)$, which is a vector tangent to I. This is compatible with pushforwards in the sense that if $\chi = f \circ \gamma$ is the pushforward of a curve via $f : M \to N$, then $\dot{\chi} = df(\dot{\gamma})$.

Definition 7. An INTEGRAL CURVE of a vector field $V \in \mathscr{X}(M)$ is a curve γ in M such that $V_{\gamma(t)} = \dot{\gamma}(t)$; i.e. V restricted to the image of γ is simply the velocity vector field of γ. A vector field V is COMPLETE if for every point $p \in M$ there is an integral curve of V passing through p that is defined on all of $\mathbb{R}$.

The FLOW of a complete vector field V is the smooth map $\phi_V : M \times \mathbb{R} \to M$ given by $\phi_V(p,t) = \gamma_{V,p}(t)$ where $\gamma_{V,p}$ is the integral curve of V with $\gamma(0) = p$. For vector fields with integral curves only defined on $t \in (-\epsilon_p, \epsilon_p)$, we can define a local flow where the domain of t is restricted based on p. We usually write $\Phi_t^V(p) = \phi_V(p,t)$, so that Φ_t^V is a family of diffeomorphisms parametrised by t.

1.3 Tensors and Tensor Fields

Definition 8. A TENSOR A of type (r,s) over an R-module V is a multilinear function

$$A : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to R.$$

The space of all (r,s)-tensors over V is denoted $T_s^r V$, and is itself an R-module under the obvious componentwise operations; if V is free with dimension n, then $T_s^r V$ has dimension n^{r+s}. Some familiar real tensor spaces are $T_0^0 V = \mathbb{R}$, $T_1^0 V = V^*$, $T_0^1 V = V$ and $T_1^1 V = L(V,V)$.

The TENSOR PRODUCT over V is a map $\otimes : T_s^r V \times T_{s'}^{r'} V \to T_{s+s'}^{r+r'}$ given by

$$(A \otimes B)\left(\theta^1 \ldots \theta^r, \vartheta^1 \ldots \vartheta^{r'}, v_1 \ldots v_s, w_1 \ldots w_{s'}\right) = A\left(\theta^1 \ldots \theta^r, v_1 \ldots v_s\right) B\left(\vartheta^1 \ldots \vartheta^{r'}, w_1 \ldots w_{s'}\right).$$

If V is a finite-dimensional real vector space and $e_1 \ldots e_n$ is a basis for V with dual basis $\theta^1 \ldots \theta^n$, then

$$\left\{ e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s} : 1 \le i_k \le n,\ 1 \le j_k \le n \right\}$$

forms a basis for $T_s^r V$.

Tensors of type $(r,0)$ are called CONTRAVARIANT, and tensors of type $(0,s)$ are COVARIANT. Tensors with r,s both non-zero are sometimes called MIXED-TYPE.

The WEDGE PRODUCT over V is the antisymmetrisation of the tensor product on covariant tensors, given by

$$A \wedge B = A \otimes B - B \otimes A.$$

Definition 9. The (r,s)-type TENSOR BUNDLE over M is formed much the same way as the tangent and cotangent bundles:

$$T^r_s M = \bigsqcup_{p\in M} T^r_s (T_p M).$$

$T^r_s M$ is an $m(1+r+s)$-dimensional manifold. An (r,s)-tensor field is a smooth section of $T^r_s M$; the space of all (r,s) tensor fields on M is denoted $\mathscr{T}^r_s M$ or $\Gamma(T^r_s M)$. The space $\mathscr{T}^r_s M$ is an $\mathscr{F}(M)$-module, and in fact it is exactly the tensor module $T^r_s(\mathscr{X}(M))$; so instead of thinking about smooth functions that are tensors over T_pM at each point, one can equivalently think about single tensors over the algebra $\mathscr{X}(M)$.

The bases we already have for $\mathscr{X}(M)$ and $\mathscr{X}^*(M)$ therefore give us a basis for $\mathscr{T}^r_s(M)$:

$$\mathscr{T}^r_s(M) = \operatorname{span}_{\mathscr{F}(M)}\left\{\partial_{i_1}\otimes\cdots\otimes\partial_{i_r}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s} : 1\le i_k\le n,\, 1\le j_k\le n\right\}$$

Definition 10. The PULLBACK of a covariant tensor field $A\in\mathscr{T}^0_s(N)$ by the smooth function $f: M\to N$ is the tensor field $f^*A\in\mathscr{T}^0_s(M)$ defined by

$$f^*A(X_1,\dots,X_s) = A(df(X_1),\dots,df(X_s)).$$

Definition 11. A CONTRACTION is a $\mathscr{F}(M)$-linear function $C^i_j : \mathscr{T}^r_s \to \mathscr{T}^{r-1}_{s-1}$ given by

$$C^i_j\left(A^{k_1\dots k_r}_{l_1\dots l_s}\right) = \sum_{\mu=1}^{n} A^{k_1\dots k_{i-1},\mu,k_{i+1}\dots k_r}_{l_1\dots l_{j-1},\mu,l_{j+1}\dots l_s}$$

where the basis vectors ∂_k, dx^l have been omitted. Contractions of tensors are a very common operation, but are difficult to convey clearly in coordinate-free notation; so we often use the index notation (i.e. writing equations of tensors in terms of their components) and the Einstein summation convention for contractions: whenever an index is repeated, once up and once down, it is implicitly summed over $\{1\dots n\}$. These components may be with respect to the coordinate frame $\{\partial_i\}$ or alternatively with repect to some more convenient basis $\{E_i\}$ of vector fields, known as a FRAME.

Some useful shortcuts in index notation are the symmetrisation

$$A^{(i_1 i_2\dots i_n)} := \frac{1}{n!}\sum_{\sigma\in S_n} A^{i_{\sigma 1} i_{\sigma 2}\dots i_{\sigma n}}$$

and the antisymmetrisation

$$A^{[i_1 i_2\dots i_n]} := \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}\sigma\cdot A^{i_{\sigma 1} i_{\sigma 2}\dots i_{\sigma n}}$$

where S_n is the symmetric group of order n. We can decompose any contravariant or covariant tensor into symmetric and antisymmetric parts.

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Definition 12. A TENSOR DERIVATION D on a manifold M is a collection of maps

$$D^r_s : \mathscr{T}^r_s M \to \mathscr{T}^r_s M$$

for every $r,s\ge 0$ such that for any tensor fields A,B on M, we have $D(A\otimes B) = DA\otimes B + A\otimes DB$ and $D(CA) = C(DA)$ for any contraction C. From this we see that we have a "product rule"

$$\begin{aligned} D\left(A\left(\theta^1,\dots,\theta^r,X_1\dots,X_s\right)\right) &= (DA)\left(\theta^1,\dots,\theta^r,X_1,\dots,X_s\right)\\ &+\sum_{i=1}^{r} A\left(\theta^1,\dots,D\theta^i,\dots,\theta^r,X_1,\dots,X_s\right)\\ &+\sum_{i=1}^{s} A\left(\theta^1,\dots,\theta^r,X_1,\dots,DX_i,\dots,X_s\right)\end{aligned}$$

and so we can write DA in terms of D's action on functions and vector fields alone. Given a vector field V and an $\mathbb{R}$-linear function $\delta : \mathscr{X}(M) \to \mathscr{X}(M)$ such that $\delta(fX) = f\delta(X) + (Vf)X$, one can therefore use the conditions above to construct a unique tensor derivation such that $D^0_0 = V$ and $D^1_0 = \delta$.

Definition 13. The LIE DERIVATIVE L is the tensor derivation generated by the Lie bracket; i.e. the unique tensor derivation satisfying

$$L_X f = df,\ L_X Y = [X, Y].$$

An alternative characterisation of the Lie derivative is the rate of change of a tensor as one flows it along the given vector field:

$$L_X A = \frac{d}{dt}\bigg|_{t=0} \left(\Phi^X_t\right)^* A. \tag{1.1}$$

This formula is useful for computing the rate of change of integrals, where the pullback of tensor fields appears naturally via the change of variables formula.

1.4 Differential Forms and Integration

On open subsets of $\mathbb{R}^n$, we have a natural notion of integration provided by the Lebesgue measure; and similarly for surfaces we have the Hausdorff measure. We can generalise integration to smooth manifolds, but we need some extra structure to do so. If we attempt to define the integral via

$$\int_\Omega f := \int_{\xi(\Omega)} f \circ \xi^{-1}\, dx^1 \cdots dx^n$$

for a chart ξ, we quickly see that this depends on the choice of chart (e.g. by simply scaling up the chart image we would increase the integral). Therefore the natural things to integrate are not functions, but objects that transform under pullbacks to cancel out the effect of changing the chart geometry (i.e. scale with the Jacobian determinant). It turns out that these objects are the completely antisymmetric covariant tensors, known as *differential forms*.

Definition 14. A DIFFERENTIAL k-FORM (or just k-form) on M is an antisymmetric tensor field $\omega \in \mathscr{T}^0_k M$; i.e. $\omega_{(i_1 \ldots i_n)} = 0$ or equivalently $\omega_{[i_1 \ldots i_n]} = \omega_{i_1 \ldots i_n}$.

The space of k-forms on M is denoted by $\Omega^k(M)$. At each point $p \in M$, the space of antisymmetric tensors at p is witten $\Lambda^k(T_p M)$. It is not hard to show that $\Lambda^k(T_p M)$ has dimension $\binom{n}{k}$ over $\mathbb{R}$, so $\Omega^k(M)$ is a module of rank $\binom{n}{k}$ over $\mathscr{F}(M)$. We write $\Omega(M) = \bigcup_{k=0}^n \Omega^k(M)$ for the space of all differential forms on M (where 0-forms are just smooth functions). The motivation for this definition is that $\Lambda^n(\mathbb{R}^n) = \operatorname{span}\{\det\}$ where the determinant is considered as a function of n vectors in $\mathbb{R}^n$ (either the rows or the columns of the usual matrix input).

We can now define the integral of a differential form, using forms on Euclidean space as a starting point and various desired properties to extend our definition to the general case. For an n-dimensional manifold, the n-forms are a one-dimensional space with basis $\left\{dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right\}$.

Definition 15. The INTEGRAL of a differential n-form $\omega = f dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ over a an open subset Ω of Euclidean n-space is given by

$$\int_\Omega f dx^1 \wedge \cdots \wedge dx^n := \overset{n}{\underset{\Omega}{\int \cdots \int}} f dx^1 \cdots dx^n \tag{1.2}$$

where the integral on the right is the usual integral of functions on $\mathbb{R}^n$ and x^μ is the usual Euclidean coordinate system. We now require that the integral is invariant under pullbacks: for a smooth map ϕ between manifolds, we require

$$\int_{\phi(\Omega)} \omega = \int_\Omega \phi^* \omega. \tag{1.3}$$

This is the CHANGE OF VARIABLES FORMULA in terms of differential forms. So now for any set $\Omega \subset M$ contained entirely within the domain of a chart ψ, we have

$$\int_{\Omega} \omega = \int_{\psi(\Omega)} \left(\psi^{-1}\right)^* \omega \tag{1.4}$$

where the integral on the right can be evaluated using Equation 1.2. For a general n-form $\omega \in \Omega^n(M)$ with compact support, take a smooth partition of unity $\{\chi_i, i \in I\}$ such that for each i, $\mathrm{supp}\chi_i \subset \mathrm{dom}\psi_i$ for some chart ψ_i. If we only require that the χ_i cover $\mathrm{supp}\omega$ then by compactness we can take I to be finite. Requiring linearity of the integral

$$\int_{\Omega} \omega + \theta = \int_{\Omega} \omega + \int_{\Omega} \theta \tag{1.5}$$

then gives

$$\begin{aligned} \int_M \omega &= \int_M \sum_{i \in I} \chi_i \omega \\ &= \sum_{i \in I} \int_{\mathrm{supp}\chi_i} \chi_i \omega. \end{aligned}$$

Each of the inner integrals is now of the form (1.4). For a k-dimensional submanifold N of M and a k-form ω on M, we simply restrict ω to TN.

The above gives us a definition of integration that works for all compactly supported forms (or alternatively allows us to integrate *any* form over a compact domain). All that one needs to remember are Equations 1.2, 1.3 and 1.5.

Definition 16. The EXTERIOR DERIVATIVE is an $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by

$$d\left(f\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_k\right) = df \wedge dz_1 \wedge dz_2 \wedge \cdots \wedge dz_k$$

and extended by linearity, where $z_1 \cdots z_k$ are any smooth functions and df is the usual differential of a function.

The Fundamental Theorem of Calculus, Divergence theorem, Green's theorem, etc are now all unified into the easily stated Stokes Theorem:

Theorem 1. ***Stokes Theorem [2].*** *For $\omega \in \Omega^n(M)$, and a region $\Omega \subset M$ whose boundary $\partial\Omega$ is a smooth submanifold, we have*

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

Definition 17. For a vector field X and k-form ω, the INTERIOR PRODUCT $i_X\omega$ is the $(k-1)$-form obtained by contraction:

$$i_X\omega\left(Y_1, \ldots, Y_{k-1}\right) = \omega\left(X, Y_1, \ldots, Y_{k-1}\right).$$

The antisymmetry of differential forms implies that $i_X \circ i_Y + i_Y \circ i_X = 0$.

The Lie derivative, exterior derivative and interior product are related by Cartan's Magic Formula

$$L_X = d \circ i_X + i_X \circ d.$$

2 Riemannian Geometry

Riemannian geometry is the study of smooth manifolds that are equipped with the additional geometric notions of distance and angle. These properties are encapsulated in a tensor field called the *metric*, which is essentially a generalisation of the inner product to non-affine spaces. Since we no longer have natural identifications between vectors based at different points, we need a separate inner product for each tangent space.

2.1 Riemannian Manifolds

Definition 18. A RIEMANNIAN METRIC g on M is a symmetric tensor field from $\mathscr{T}_2^0 M$ that is positive definite when considered as a bilinear form (that is, the eigenvalues of the corresponding map in $L(V, V^*)$ given by $v \mapsto g(v, \cdot)$ are all positive). A Riemannian manifold is a pair (M, g) where g is a Riemannian metric on the manifold M.

The Riemannian metric gives a canonical correspondence between covariant and contravariant tensor indices. When tensor indices are raised or lowered from their usual position, contraction with the metric is assumed, e.g. for a tensor $R \in \mathscr{T}_2^0(M)$,

$$R^i{}_j = g^{ik} R_{kj}.$$

Definition 19. The CANONICAL VOLUME FORM $d\mu$ (also commonly dvol, dV, vol_n) on a Riemannian n-manifold (M, g) is the n-form defined by

$$d\mu(v_1 \cdots v_n) = \sqrt{\det_{ij} g(v_i, v_j)}$$

and in coordinates has the form

$$d\mu = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n.$$

On Euclidean space, integration with repect to the canonical volume form agrees with the usual integration, hence the notation $d\mu$. This volume form gives us a natural way to integrate functions.

2.2 Connections and Covariant Differentiation

A CONNECTION D on a manifold M is a function

$$D : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M) : V, W \mapsto D_V W$$

that is $\mathscr{F}(M)$-linear in V, $\mathbb{R}$-linear in W and such that $D_V(fW) = (Vf)W + fD_V W$ for $f \in \mathscr{F}(M)$. $D_V W$ is the COVARIANT DERIVATIVE of W with respect to V for the connection D. The pair (V, D_V) uniquely specify a tensor derivation; the notation D_V will be used in general for the tensor derivation (e.g. for a scalar field f, we write $D_V f = Vf$).

Theorem. ***The Fundamental Theorem of Riemannian Geometry.*** *On a Riemannian manifold (M, g), there exists a unique connection ∇ such that $[V, W] = \nabla_V W - \nabla_W V$ and $\nabla_X g = 0$.*

The connection ∇ appearing above is the LEVI-CIVITA CONNECTION, and the tensor derivation associated with its covariant derivative gives a canonical way to differentiate tensor fields on (M, g). The first condition is referred to as the TORSION-FREE condition, and the second as the METRIC-COMPATIBLE condition. The metric-compatibility can be seen as generalising the identity

$$\partial_i (X \cdot Y) = (\partial_i X) \cdot Y + X \cdot (\partial_i Y)$$

for the partial derivative and dot product on $\mathbb{R}^n$.

Definition 20. The COVARIANT DIFFERENTIAL of a tensor field $A \in \mathscr{T}^r_s M$ is the tensor $\nabla A \in \mathscr{T}^r_{s+1}$ defined by

$$(\nabla A)\left(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, V\right) = (\nabla_V A)\left(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s\right).$$

We write ∇_i for ∇_{∂_i}.

Definition 21. The CHRISTOFFEL SYMBOLS are defined by ${\Gamma^i}_{jk} = dx^i\left(\nabla_j \partial_k\right)$ and can be calculated from the metric as

$${\Gamma^i}_{jk} = \frac{1}{2} g^{im}\left(\partial_k g_{mj} + \partial_j g_{km} - \partial_m g_{jk}\right).$$

Definition 22. The DIVERGENCE of a vector field X is

$$\mathrm{div} X = \mathrm{tr} \nabla X = dx^j\left(\nabla_j X\right).$$

An important formula that holds for all vector fields X is

$$L_X d\mu = \mathrm{div} X \, d\mu.$$

2.3 Intrinsic Curvature

Curvature is a measurement of the local deviation of the geometry of a space from that of $\mathbb{R}^n$. We start with a fairly complicated tensor, and then define more approachable quantities derived from it. In index notation, the three common curvature tensors are all notated by R, with the number of indexes distinguishing them.

Definition 23. The RIEMANN CURVATURE TENSOR of a Riemannian manifold is the $(1,3)$-tensor field (usually thought of as a map $V \times V \times V \to V$) given by

$$R(U,V)W = \left(\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]}\right) W.$$

One The "totally covariant" curvature tensor is the $(0,4)$-tensor field

$$R(U,V,X,W) = g(R(U,V)W, X).$$

In co-ordinates, the curvature tensor is

$${R^i}_{jkl} = \partial_l \Gamma^i_{kj} - \partial_k \Gamma^i_{lj} + \Gamma^i_{lm} \Gamma^m_{kj} - \Gamma^i_{km} \Gamma^m_{lj}$$

where the components are ordered such that ${R^i}_{jkl} = dx^i\left(R\left(\partial_k, \partial_l\right) \partial_j\right)$.

One sees that for Euclidean space (where the Christoffel symbols vanish), we have $\nabla_i \nabla_j X = \nabla_j \nabla_i X$ and $[\partial_i, \partial_j] = 0$; so the Riemann tensor vanishes (and so will all the curvature measurements we define in the following).

Definition 24. The RICCI CURVATURE is the $(0,2)$-tensor

$$\mathrm{Ric}(W,V) = \mathrm{tr}\left(U \mapsto R(U,V)W\right)$$

or in co-ordinates $R_{ij} = {R^k}_{ikj}$.

The Ricci curvature can be interpreted as measuring deviation in the volume of thin cones - the larger the value of $\mathrm{Ric}(v,v)$, the smaller the volume of a thin cone in the direction v with a given length. When $\mathrm{Ric}(v,v) = 0$, the volume of the cone will agree with one in Euclidean space with the same solid angle and length to first order in the length. One can formalise this in terms of inequalities involving the canonical volume form; see e.g. [3, Ch 9].

Definition 25. The scalar curvature is the trace of the Ricci tensor with respect to the metric:

$$\mathrm{Sc} = R = \mathrm{tr}_g \mathrm{Ric} = g^{ij} R_{ij}.$$

The scalar curvature at p measures the deviation in the volume of a small ball around p: The greater the scalar curvature, the smaller the volume of a small ball. A scalar curvature of 0 at p means the volume of an ϵ-ball agrees with that in Euclidean space up to 4th order in p.

Definition 26. The SECTIONAL CURVATURE of a plane $\Pi = \mathrm{span}\,(u, v) \subset T_pM$ is

$$\sec(\Pi) = \frac{R_p(u, v, u, v)}{g_p(u, u)\, g_p(v, v) - g_p(u, v)^2}$$

and depends only upon the plane Π, not the basis $\{u, v\}$ chosen. If every sectional curvature at p is zero, then R_p is zero.

The sectional curvature has an interpretation in terms of the Gaussian curvature of submanifolds with the given tangent plane; we will see this later.

2.4 Geodesics

Definition 27. A tensor field A along a curve γ is PARALLEL if $\nabla_{\dot\gamma} A = 0$. A curve γ is a GEODESIC if its velocity vector is parallel; i.e. $\nabla_{\dot\gamma}\dot\gamma = 0$. In co-ordinates, this condition becomes the GEODESIC EQUATION:

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk}\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 0$$

where γ^i is shorthand for the map $x^i \circ \gamma : I \to \mathbb{R}$. Geodesics have constant speed, because $\nabla_{\dot\gamma} g(\dot\gamma, \dot\gamma) = 2g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = 0$. The geodesic equation is second-order; so an initial point $\gamma(0)$ and direction $\dot\gamma(0)$ specify a unique geodesic; if all the geodesics can be defined on all of $\mathbb{R}$, the manifold is GEODESICALLY COMPLETE.

Definition 28. The EXPONENTIAL MAP at p is a map $\exp_p : T_pM \to M$ defined by

$$\exp_p(u) = \gamma_u(1)$$

where γ_u is the unique geodesic starting at p with velocity u. Intuitively, the exponential map wraps the tangent space around the manifold.

Theorem 2. ***Hopf-Rinow [1].*** *For a connected Riemannian manifold (M, g), the following are equivalent:*

1. Every closed, bounded subset of M is compact;
2. (M, d) is complete as a metric space, where the metric is $d(p, q) = \sup\{L(\gamma) : \gamma \text{joins } p \text{to } q\}$ where $L(\gamma) = \int_\gamma \sqrt{g(\dot\gamma, \dot\gamma)}$ is the length of γ.
3. M is geodesically complete.

2.5 Geometry of Submanifolds

Definition 29. A HYPERSURFACE of a $n+1$-manifold (M, g) is a submanifold N of dimension n.

At each point p, there is a natural vector space inclusion $T_pN \subset T_pM$; so we can define the one-dimensional orthogonal complement $T_pN^\perp \subset T_pM$. There is therefore a unique (up to sign) unit normal to T_pN for each point; if N is orientable, we can define a global vector field of normals

$$n : N \to TM$$
$$p \mapsto n(p) \in T_pN^\perp.$$

Definition 30. The SHAPE TENSOR of a hypersurface $N \subset M$ is the $(1,1)$-tensor

$$S = \nabla \nu$$

where ν is the "outwards" normal to N (in some cases this can be defined sensibly; in general, simply take a non-vanishing normal vector field and declare it to be "outwards"). The SECOND FUNDAMENTAL FORM of M is simply the $(0,2)$-tensor obtained by lowering an index of S:

$$k(X,Y) = g(S(X), Y) = g(\nabla_X \nu, Y),$$

sometimes written $\mathrm{II}(X,Y)$. k is symmetric, and supplies the normal component of the covariant derivative: for $X, Y \in \mathscr{X}(M)$, we have

$$\nabla_X Y = D_X Y + k(X,Y)\,\nu$$

where D is the Levi-Civita connection on the Riemannian submanifold $(N, h = g|_{TN})$, and the vector fields are extended to $\mathscr{X}(M)$.

Theorem. ***Gauss Equation.*** *For a hypersurface $N \subset M$ and vector fields X, Y, Z, W on N, we have*

$$R_N(X,Y,Z,W) = R_M(X,Y,Z,W) + k(X,Z)\,k(Y,W) - k(X,W)\,k(Y,Z)$$

where R_M, R_N are the Riemann curvature tensors of (M,g), (N,h) respectively.

Codazzi Equation. *For a hypersurface $N \subset M$ and vector fields X, Y, Z on N, we have*

$$R_M(X,Y,\nu,Z) = \nabla_X k(Y,Z) - \nabla_Y k(X,Z).$$

The shape tensor embodies the *extrinsic* curvature of the submanifold N; i.e. it depends on the particular embedding $N \to M$ and gives us geometric information about the embedding.

Definition 31. The PRINCIPAL CURVATURES of a submanifold $N \subset M$ are the eigenvalues of the shape tensor.

The MEAN CURVATURE is the mean of the principal curvatures, $H := \frac{1}{n}\mathrm{tr} S$.

The Gaussian curvature of a two-dimensional submanifold N in a Riemannian 3-manifold can be found using the Gauss equation in an orthonormal frame $\{E_1, E_2\}$ for $\mathscr{X}(N)$:

$$K = R_{N\,1212} = R_{M\,1212} + k_{11}k_{22} - k_{12}k_{21} = \sec TN + \det S.$$

3 Partial Differential Equations

3.1 Calculus of Variations

One very useful approach to partial differential equations (PDEs) is via the calculus of variations and functional analysis. In the calculus of variations, one studies functionals (also known as ACTIONS) of the form

$$S_{\mathscr{L}}^{U} : \mathscr{A} \to \mathbb{R} : u \mapsto \int_U \mathscr{L}\left(u\left(x\right), \nabla u\left(x\right)\right) dx$$

where $\mathscr{L} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the LAGRANGIAN, U is the domain of interest and $\mathscr{A} \subset \mathbb{R}^U$ is a space of functions. Varying u and integrating by parts then shows that for any *smooth* u that locally extremises the action in the space $C^\infty\left(U\right)$, the Euler-Langrange equations

$$\operatorname{div} \frac{\partial \mathscr{L}}{\partial \nabla u} = \frac{\partial \mathscr{L}}{\partial u} \tag{3.1}$$

hold. Given a PDE, one then attempts to find a Lagrangian $\mathscr{L}$ such that the Euler-Lagrange equation is equivalent to the PDE. One can then study the set of local minimisers of the action $S_{\mathscr{L}}$, which are known as WEAK SOLUTIONS. If one can prove the existence of a weak solution and then show that it is in fact smooth (or REGULAR), then one has in fact shown the existence of a strong solution. This is done by adding some structure to $\mathscr{A}$: if we can make $\mathscr{A}$ a compact topological space such that the action functional is continuous, then we have a minimiser. (In practice we often do not have full continuity, but a weaker and still sufficient condition). Usually $\mathscr{A}$ will in fact be a Banach space, such as a Sobolev or Hölder space.

3.2 Functional Analysis

The natural setting for the calculus of variations is functional analysis, and in particular the theory of Banach and Hilbert spaces. We will briefly define some useful categories of vector spaces and state some useful theorems.

Definition 32. A TOPOLOGICAL VECTOR SPACE (over $\mathbb{R}$) is a real vector space V equipped with a topology such that scalar multiplication and addition are continuous.

Definition 33. A NORMED VECTOR SPACE (over $\mathbb{R}$) is a vector space V equipped with a function $\|\cdot\| : V \to \mathbb{R}$ such that

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|\alpha x\| = |\alpha| \|x\|$ for any $\alpha \in \mathbb{R}, x \in V$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Any normed vector space is a metric space with metric $d\left(x, y\right) = \|x - y\|$, and is a topological vector space with the metric topology.

Definition 34. A BANACH SPACE is a normed vector space which is complete as a metric space.

Definition 35. A HILBERT SPACE is a Banach space whose norm is derived from an inner product (i.e. there exists a positive definite symmetric bilinear form B on V such that $\forall x \; \|x\|^2 = B\left(x, x\right)$).

Definition 36. The DUAL of a topological vector space V is the space of continuous linear functionals on V:

$$V^* := C\left(V, \mathbb{R}\right) \cap L\left(V, \mathbb{R}\right).$$

The dual space is a Banach space when given the operator norm

$$\|\xi\|_{V^*} := \sup_{\|x\|_V=1} |\xi(x)|.$$

There is a natural embedding i of a topological vector space V into its double dual V^{**}, given by

$$(i(x))(\xi) = \xi(x).$$

In the finite dimensional case, this is an isomorphism; but it is just an inclusion in general. If i is an isomorphism then V is called REFLEXIVE.

Definition 37. The WEAK TOPOLOGY on a topological vector space V is the coarsest topology such that every (strongly) continuous linear functional is (weakly) continuous. If $x_n \to x$ in the weak topology (i.e. $\xi x_n \to \xi x$ for every $\xi \in V^*$), we write $x_n \rightharpoonup x$.

Definition 38. The WEAK-* TOPOLOGY on the dual space V^* is the coarsest topology such that the maps $i(x) : V^* \to \mathbb{R}$ are continuous for each $x \in V$.

Theorem 3. *(Banach-Alaoglu). The closed unit ball* $\left\{x \in V^* \middle| \|x\| \leq 1\right\}$ *is compact in the weak-* topology.*

Corollary. *In a reflexive Banach space (e.g. a Hilbert space), every bounded sequence has a weakly convergent subsequence.*

Proposition 1. *Closed, convex subsets of a Hilbert space are weakly closed.*

3.3 Function spaces

The Banach spaces we will be using are spaces of functions on subsets of $\mathbb{R}^n$ (or more generally on manifolds). The simplest examples are integrable functions (the L^p spaces) and k-times continuously differentiable functions (the C^k spaces).

We will define a number of spaces of functions in the following sections. In general, when $\mathscr{A}(U)$ is a space of real-valued functions over U, $\mathscr{A}(U, \mathbb{R}^n)$ will denote the space of vector fields $U \to \mathbb{R}^n$ such that each component function is in $\mathscr{A}(U)$, and the norm can be taken as

$$\|X\|_{\mathscr{A}(U,\mathbb{R}^n)} = \| |X| \|_{\mathscr{A}(U)}$$

for any norm $|\cdot|$ on $\mathbb{R}^n$. (It turns out that all such norms are topologically equivalent.)

$\mathscr{A}_0(U)$ denotes the functions in $\mathscr{A}(U)$ with compact support. The local version of $\mathscr{A}(U)$ is defined by

$$\mathscr{A}_{\text{loc}}(U) = \left\{f : U \to \mathbb{R} \middle| f|_K \in \mathscr{A}(K) \text{ for each compact } K \subset U\right\}.$$

3.3.1 L^p spaces

Definition 39. The L^p NORM of a function $f : U \to \mathbb{R}$ is

$$\|f\|_p = \|f\|_{L^p(U)} = \left(\int_U |f|^p\right)^{1/p}$$

or for $p = \infty$,

$$\|f\|_p = \text{ess sup}_{x\in U} |f(x)|.$$

The L^p SPACE over U is

$$L^p(U) = \left\{f : U \to \mathbb{R} \middle| \|f\|_p < \infty\right\}$$

where we identify functions that agree almost everywhere (i.e. except on a set of Lebesgue measure zero). For $p = 2$, we obtain a Hilbert space $L^2(U)$ with the inner product

$$\langle f, g\rangle = \int_U fg.$$

3.3.2 C^k spaces

Definition 40. The space of k-times continuously differentiable functions on a domain U is denoted by $C^k(U)$. When f and its derivative are bounded (or alternatively when U is compact, which guarantees boundedness), we can make $C^k(U)$ a normed space.

The C^k NORM of a function $f : \mathbb{R}^n \supset U \to \mathbb{R}$ is defined by

$$\|f\|_{C^k} = \sum_{j=0}^{k} \sum_{\alpha \in n^j} \sup |D_\alpha f|$$

where

$$D_\alpha = \partial_{\alpha(1)} \partial_{\alpha(2)} \cdots \partial_{\alpha(j)}$$

is the j^{th}-order derivative operator described by the MULTI-INDEX $\alpha \in n^j = \{1..n\}^j$ and $D_\emptyset f = f$.

The sum over all derivatives will appear again later in the context of other function spaces; it essentially means that convergence of a function requires (some simpler) convergence of all of its derivatives up to the relevant order.

3.3.3 Weak Derivatives and Sobolev Spaces $W^{k,p}, H_k$

To apply the tools of functional analysis to the solution of PDEs, we would like to work with a complete space of functions. The Sobolev spaces are the completion of the C^k spaces with respect to norms that ensure L^p convergence of the first k derivatives.

If $f \in C^1(U, \mathbb{R})$, then we have a gradient field $\nabla f \in C^0(U, \mathbb{R}^n)$. For any $Y \in C_0^\infty(U, \mathbb{R}^n)$ integration by parts gives

$$\int_U (\nabla f \cdot Y + f \text{div} Y) = \int_U \text{div}(fY) = 0.$$

For non-differentiable f, we use this as the *definition* of ∇f:

Definition 41. $X \in L^1_{\text{loc}}(U, \mathbb{R}^n)$ is the WEAK GRADIENT of $f \in L^1_{\text{loc}}(U)$ if for each $Y \in C_0^\infty(U, \mathbb{R}^n)$ we have

$$\int_U (X \cdot Y + f \text{div} Y) = 0.$$

If f has a weak gradient, it is unique and we call it ∇f.

Definition 42. The SOBOLEV SPACE $W^{1,p}$ is

$$W^{1,p}(U) = \left\{ f \in L^p(U) \middle| f \text{ is weakly differentiable, } \nabla f \in L^p(U, \mathbb{R}^n) \right\}.$$

The $(1,p)$ SOBOLEV NORM of a weakly differentiable function f is

$$\|f\|_{W^{1,p}(U)} = \left(\|f\|^p_{L^p(U)} + \|\nabla f\|^p_{L^p(U,\mathbb{R}^n)} \right)^{1/p}.$$

For $p = 2$, $H_1(U) = W^{1,2}(U)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{H_1(U)} = \langle f, g \rangle_{L_2(U)} + \int_U \nabla f \cdot \nabla g.$$

We can generalise the Sobolev spaces to higher order derivatives by iterating integration by parts, which gives

$$\int_U \left(f D_\alpha \varphi - (-1)^k \varphi D_\alpha f \right) = 0$$

for any $\varphi \in C_0^\infty(U)$ and multi-index α of order k. We then say

Definition 43. A function h is an αth weak partial derivative of f if for every $\varphi \in C_0^\infty(U)$ we have

$$\int_U \left(f D_\alpha \varphi - (-1)^k \varphi h \right) = 0.$$

If f has an αth weak partial derivative, it is unique and we call it $D_\alpha f$. We then obtain the higher-order (k,p) Sobolev spaces using the norms

$$\|f\|_{W^{k,p}(U)}^p = \sum_{j=0}^{k} \sum_{\alpha \in n^j} \|D_\alpha f\|_{L_p(U)}^p$$

which for $p = 2$ are Hilbert spaces $H_k(U)$ with inner products

$$\langle f, g \rangle_{H_k(U)} = \sum_{j=0}^{k} \sum_{\alpha \in n^j} \langle D_\alpha f, D_\alpha g \rangle_{L_2(U)}.$$

$W_0^{k,p}(U)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(U)$, and is exactly the set of $W^{k,p}$ functions vanishing on $\partial\Omega$.

Theorem 4. *The smooth subspace $C^\infty(U) \cap W^{k,p}(U)$ is dense in U.*

In particular, we have $C_0^\infty(U) \ni u * \varphi_n \to u \in W_0^{k,p}(U)$ where φ_n is an approximation of the identity and $*$ denotes convolution.

3.4 The Direct Method

While it would be convenient for the purposes of minimisation to have a continuous functional over a compact space, it is usually not quite that simple. We give here an example of a slightly more subtle minimisation argument, a specific case of what is known as the *direct method* of the calculus of variations. The proof is immediate given basic Banach space theory; in applications, all the work lies in showing the desired properties hold for the given set. The combination of coerciveness and weak closure combine to give weak convergence of the minimising subsequence.

Theorem 5. *Let $\mathscr{H}$ be a separable Hilbert space (e.g. $\mathscr{H}_1$). Then if a non-empty subset $K \subset \mathscr{H}$ is closed in the weak topology and $F : K \to \mathbb{R}_{\geq 0}$ is*

1. *weakly lower semi-continuous: $F(x) \leq \liminf_{k\to\infty} F(x_k)$ when $x_k \rightharpoonup x$; and*
2. *coercive: $F(x) \to \infty$ whenever $\|x\|_{\mathscr{H}} \to \infty$*

then F has a minimum on K.

Proof. Take a sequence x_n such that $F(x_n) \to \inf_K F$. Then x_n is bounded (if not, there would be a subsequence on which $\|x_n\| \to \infty$ but $F(x_n) \to \inf_K F \neq \infty$). Since $\mathscr{H}$ is reflexive, x_n has a weakly convergent subsequence $x_{n_k} \rightharpoonup x_0$ by the Corollary to Theorem 3. We therefore have $F(x_0) \leq \liminf_{k\to\infty} F(x_{n_k}) = \lim_{n\to\infty} F(x_n) = \inf_K F$; i.e. x_0 minimises F over K. □

We will apply this theorem to Laplace's equation in the next subsection, and Plateau's problem in Section 4.

3.5 Laplace's Equation

We will now apply the preceding theory to Laplace's equation: $\Delta u = 0$. We will use Dirichlet boundary conditions; i.e. prescribed values for u on the boundary of the compact domain of interest U.

Laplace's equation is easily seen to be the Euler-Lagrange equation of the Lagrangian

$$\mathscr{L}(u, \nabla u) = |\nabla u|^2 .$$

The *weakly harmonic* functions are then the local minimisers of the action $E(u) = \|\nabla u\|_{L^2}^2$ associated with this Lagrangian; i.e. the $u \in H_1(U)$ such that

$$\left.\frac{d}{dt}\right|_{t=0} E(u + t\varphi) = 2\int_U \nabla u \cdot \nabla \varphi = 0$$

for all $\varphi \in C_0^\infty(U)$. Given a $v \in H_1(U)$ (we are only really interested in the values $v|_{\partial U}$, but this makes the analysis easier), $u \in H_1(U)$ is a weak solution of Laplace's equation on U with Dirichlet boundary condition v if u locally minimises E and $u - v \in W_0^{1,2}(U)$. We denote the space of functions meeting the boundary conditions by

$$\mathscr{H}_v = \left\{ u \in H_1(U) \,\middle|\, u - v \in W_0^{1,2}(U) \right\}.$$

3.5.1 Existence of Weak Solution

Lemma. *(Poincaré Inequality). For $U, \mathscr{H}_v$ as above, there is a constant C such that for every $u \in \mathscr{H}_v$ we have*

$$\|u\|_{L^2}^2 \leq C\left(E(u) + \|v\|_{H_1}^2\right)$$

Theorem 6. *Given a boundary condition $v \in H_1(U)$, there exists a $u_0 \in \mathscr{H}_v$ that minimises the functional E.*

Proof. First, note that $\mathscr{H}_v$ is non-empty because $v \in \mathscr{H}_v$. If $u_n \in \mathscr{H}_v$ converges to $u \in H_1(U)$, then we have

$$u - v = \lim_{n\to\infty}(u_n - v) \in W_0^{1,2}(U)$$

because $W_0^{1,2}(U)$ is closed by definition; i.e. $\mathscr{H}_v$ is closed. If $u, w \in \mathscr{H}_v$ and $t \in [0,1]$, then for a convex combination we have

$$tu + (1-t)w - v = t(u - v) + (1-t)(w - v) \in W_0^{1,2}(U)$$

because both $u - v$ and $w - v$ are in $W_0^{1,2}(U)$; i.e. $\mathscr{H}_v$ is convex. By Proposition 1, this implies $\mathscr{H}_v$ is weakly closed. If $u_n \rightharpoonup u \in H_1(U)$, we have

$$\begin{aligned} E(u) &= \lim_{n\to\infty}\int_U \nabla u_n \cdot \nabla u \\ &\leq \|\nabla u\|_{L^2} \liminf_{n\to\infty} \|\nabla u_n\| \\ &= \sqrt{E(u)} \liminf_{n\to\infty} \sqrt{E(u_n)} \end{aligned}$$

and therefore E is weakly lower semi-continuous. Finally, if $\|u_n\| \to \infty$ for $u_n \in \mathscr{H}_v$, then either $\|u_n\|_{L^2} \to \infty$ or $E(u_n) \to \infty$. But the Lemma above means the first would imply the second; so no matter what we have $E(u_n) \to \infty$; i.e. E is coercive. We have shown $\mathscr{H}_v$ and E satisfy all the requirements of Theorem 5; so there exists a minimiser. □

3.5.2 Elliptic Regularity

Now that we have a weak solution, we want to show that it is regular and therefore a strong solution. For Laplace's equation (and indeed linear elliptic PDEs in general), regularity is provided by the Sobolev Embedding Theorem, which we state (a weakened version of) without proof (see e.g. [5, Ch 5.6]).

Theorem 7. *(Sobolev Embedding Theorem) For a bounded domain $U \subset \mathbb{R}^n$ and $u \in W^{k,p}(U)$:*

Case 1. $k < n/p$: *We have $u \in L^q(U)$ where $q = np/(n - kp)$.*

Case 2. $k > n/p$: *We have $u \in C(U)$.*

Lemma. *If $u \in H_1(U)$ is weakly harmonic, then $u \in H_k(U)$ for every k.*

Proof. Let $w = \partial_i u \in L^2(U)$. For any $\varphi \in C_0^\infty(U)$, define $\psi = \partial_i \varphi$; then $\psi \in C_0^\infty(U)$ and we have (integrating by parts twice)

$$\begin{aligned}\int_U w\Delta\varphi &= \int_U (\partial_i u)(\Delta\varphi) \\ &= -\int_U u\partial_i\Delta\varphi \\ &= -\int_U u\Delta\partial_i\varphi \\ &= -\int_U u\Delta\psi \\ &= \int_U \nabla u \cdot \nabla\psi = 0\end{aligned}$$

since u is weakly harmonic. This implies that $w \in H_1(U)$ [6, Prop 2.4], so applying the definition of the weak derivative, this shows that w is also weakly harmonic. Repeating the argument starting with w instead of u and inducting shows that all orders of weak partial derivatives of u exist. □

Theorem 8. *(Weyl's Lemma) Weakly harmonic functions are harmonic.*

Proof. Let u be weakly harmonic. By the lemma, $u \in H_k(U)$ for each k, so the weak partial derivatives u_{ij} exist and are in $W^{k,2}(U)$ for each k. In particular, $u_{ij} \in W^{k,2}(U)$ for some $k > n/2$, so applying Theorem 7 shows that $u_{ij} \in C(U)$ and therefore $u \in C^2(U)$. Since u is a sufficiently differentiable local minimiser of the action, it satisfies the Euler-Lagrange equation; i.e. $\Delta u = 0$.

□

4 Application: Minimal Surfaces

The study of minimal surfaces arose from looking for surfaces that minimise area amongst all "nearby" surfaces. The particular notion of "nearby" depends on the problem; a very common example is the surface minimising area amongst all surfaces with a given boundary. We can get some results from a purely differential geometric approach, but ultimately proving the existence of solutions requires the theory of PDEs.

Definition 44. The AREA of a parametrised surface $X : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is

$$A(X) = \int_U \sqrt{|X_u \times X_v|}\, du\, dv$$

where subscripts denote partial derivatives.

If the family of surfaces considered is closed under local continuous deformations of the parametrisation, then a necessary condition for area minimisation is given by the Euler-Lagrange equations for the Lagrangian $\mathscr{L} = \sqrt{|X_u \times X_v|} = \sqrt{|X_u|^2 |X_v|^2 - (X_u \cdot X_v)^2}$:

$$0 = \partial_u \left(\frac{\partial \mathscr{L}}{\partial X_u} \right) + \partial_v \left(\frac{\partial \mathscr{L}}{\partial X_v} \right) = \partial_u \left(\frac{f X_u - g X_v}{\sqrt{ef - g^2}} \right) + \partial_v \left(\frac{e X_v - g X_u}{\sqrt{ef - g^2}} \right) \tag{4.1}$$

where $e = |X_u|^2$, $f = |X_v|^2$ and $g = X_u \cdot X_v$. We use this as the definition of a minimal surface:

Definition 45. A parametrised surface $X : U \to \mathbb{R}^n$ is a MINIMAL SURFACE if it satisfies Equation 4.1.

For $n = 3$, this is equivalent to the mean curvature vanishing. In more generality, for surfaces $N \subset M$ in a Riemannian 3-manifold M, the critical points of the area functional $\int_N d\mu_N$ are still the surfaces with zero mean curvature: applying the theory of differential forms shows that

$$0 = \frac{d}{dt}\Big|_{t=0} \int_{\Phi_t^X N} d\mu_{\Phi_t^X N} = \int_N L_X d\mu_N = \int_N H g(X, \nu_N)\, d\mu_N$$

for any vector field X, so $H = 0$. We will stick to the Euclidean case for simplicity.

Definition 46. A map $X : (M, g) \to (N, h)$ between Riemannian manifolds is CONFORMAL if it preserves angles; i.e. if there is a real function λ on M such that

$$X^* h = \lambda^2 g.$$

For $X : \mathbb{R}^2 \to \mathbb{R}^n$, this can be written as

$$X_u \cdot X_v = 0 = |X_u|^2 - |X_v|^2 . \tag{4.2}$$

It is not hard to see from (4.1) and (4.2) that a conformally parametrised surface is minimal if and only if it is harmonic (i.e. $X_{uu} + X_{vv} = 0$).

4.1 Plateau's Problem

The particular problem we will discuss is known as *Plateau's Problem.* Given a simple closed curve $\Gamma \subset \mathbb{R}^n$, consider the family of disk-type surfaces spanning Γ:

$$C_\Gamma = \left\{ X \in C^0\left(\bar{D}, \mathbb{R}^n\right) \middle| X|_D \in C^2\left(D, \mathbb{R}^n\right), X\left(\partial D\right) = \Gamma \right\}$$

where $D = \left\{ (u,v) \in \mathbb{R}^2 \middle| u^2 + v^2 < 1 \right\}$ is the open unit disk. Then is there a surface of minimal area in C_Γ? Attempting to minimise A as a functional on C_Γ, we hit two issues:

- C_Γ is not a very nice space; in particular, it is not complete.
- A is invariant under reparametrisations (i.e. the group of diffeomorphisms of the disk).

To address the first issue, we instead use a larger class of weakly differentiable surfaces:

$$\mathscr{H}_\Gamma = \left\{ X \in H_1\left(D, \mathbb{R}^n\right) \middle| X|_{\partial D} \text{ is a monotone parametrisation of } \Gamma \right\}.$$

For the second, we replace the area by the Dirichlet energy.

Definition 47. The Dirichlet energy of a parametrised surface $X : \mathbb{R}^2 \supset U \to \mathbb{R}^n$ is

$$E\left(X\right) = \frac{1}{2}\int_U \left|\nabla X\right|^2 = \frac{1}{2}\int_U \left(\left|X_u\right|^2 + \left|X_v\right|^2\right) du\, dv.$$

The energy is invariant under conformal transformations of U; i.e. $E\left(X \circ \psi\right) = E\left(X\right)$ for any conformal diffeomorphism $\psi : U \to U$.

4.1.1 Existence of a Minimiser

We would like to apply Theorem 5 to find a minimiser for E; but there is still an issue:

Proposition 2. *$\mathscr{H}_\Gamma$ is not weakly closed.*

Proof. Take a sequence $\omega_j \in D$, $\omega_j \to 1$ and define the sequence of conformal diffeomorphisms

$$g_j\left(z\right) = \frac{\omega_j + z}{1 + \bar{\omega}_j z}$$

of the disc, where we have identified $D \subset \mathbb{C}$ to use the mutliplication. For each z we have pointwise convergence $g_j\left(z\right) \to 1$; so for $X \in \mathscr{H}_\Gamma$ we have $X_j = X \circ g_j \to g_j\left(1\right)$ pointwise. By the Poincaré Inequality, $\|X_j\|$ is bounded, so we can pass to a weakly convergent subsequence $X_j \rightharpoonup X_0$. Noting that $E\left(X, Y\right) = \int \nabla X \cdot \nabla Y = \frac{1}{2}\left(E\left(X+Y\right) - E\left(X\right) - E\left(Y\right)\right)$ is conformally invariant, we have

$$\begin{aligned}
\int \nabla X_0 \cdot \nabla \varphi &= \lim_{j\to\infty} \int \nabla X_j \cdot \nabla \varphi \\
&= \lim_{j\to\infty} \int \nabla X \cdot \nabla \varphi \\
&= \lim_{j\to\infty} \int \nabla X \cdot \nabla\left(\varphi \circ g_j\right) \\
&= \int \nabla X \cdot \nabla\left(\lim_{j\to\infty} \varphi \circ g_j\right) \\
&= 0
\end{aligned}$$

because φ converges weakly to the constant map 1. This implies X_0 itself is a constant map, so $X_0\left(\partial D\right) \neq \Gamma$. ☐

To circumvent this, we note that the conformal group of the disc is $\mathscr{G} = PSL(2, \mathbb{R})$, which acts three-point homogenously (assuming fixed order) on ∂D. Therefore we can choose $\theta_1, \theta_2, \theta_3 \in \partial D$ and $p_1, p_2, p_3 \in \Gamma$ and use the restricted class of maps

$$\mathscr{H}_\Gamma^* = \left\{ X \in \mathscr{H} \middle| X(\theta_k) = p_k \forall k \right\}$$

without losing any surfaces (each surface in $\mathscr{H}_\Gamma$ can be conformally reparametrised appropriately using the three-point homogeneity). Now that we have removed the invariance, $\mathscr{H}_\Gamma$ is weakly closed [7, 6].

Theorem 9. *If $\mathscr{H}_\Gamma$ is non-empty, then E achieves a minimum in $\mathscr{H}_\Gamma$.*

Proof. Since $\mathscr{H}_\Gamma$ is non-empty, we can reparametrise a surface in $\mathscr{H}_\Gamma$ to show that $\mathscr{H}_\Gamma^*$ is non-empty. E is both weakly lower-semicontinuous and coercive (proofs very similar to Subsection 3.5.1); so by Theorem 5 there is an $X_0 \in \mathscr{H}_\Gamma^*$ that minimises E. Since any surface in $\mathscr{H}$ has a parametrisation in $\mathscr{H}_\Gamma^*$, X_0 minimises E over $\mathscr{H}_\Gamma^*$. □

4.1.2 Regularity of Minimiser

Theorem 10. *The minimiser $X_0 \in \mathscr{H}_\Gamma$ is a harmonic map $X_0 \in C^2(D, \mathbb{R}^n)$.*

Proof. Since X_0 minimises E, for any $Y \in W_0^{1,2}(D)$ we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} E(X_0 + tY) \\ &= \frac{d}{dt}\Big|_{t=0} \left(E(X_0) + t^2 E(Y) + 2tE(X, Y) \right) \\ &= 2 \int \nabla X_0 \cdot \nabla Y; \end{aligned}$$

i.e. the components of X_0 are weakly harmonic. By Theorem 8, the components of X_0 are C^2 and harmonic. □

By the Riemann mapping theorem ($\psi_t D$ is conformal to D), we also have

$$0 = \frac{d}{dt}\Big|_{t=0} E\left(X_0 \circ \psi_t^{-1}; \psi_t D\right)$$

where the second argument to E is the domain of the energy integral. After some work [7, 6], this implies that X_0 is also conformal.

4.1.3 Details

We have shown that if there are *any* surfaces spanning Γ, there is a minimal one with least energy. One can show that if Γ is rectifiable (can be parametrised by a weakly differentiable function $S^1 \to \Gamma$) then $\mathscr{H}_\Gamma$ is non-empty [6]. Applying conformal uniformisation allows one to show that energy minimisation and area minimisation are equivalent [6]. Thus, we conclude with the solution to Plateau's Problem:

Theorem 11. *For a rectifiable simple closed curve $\Gamma \subset \mathbb{R}^n$, there exists a harmonically/conformally parametrised, minimal surface Ω spanning Γ that minimises area amongst all surfaces spanning Γ.*

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